

# Locally Ordinal Bayesian Incentive Compatibility

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## Abstract

We investigate the locally ordinal notion of Bayesian incentive compatibility (LOBIC) of deterministic voting mechanisms. We consider a standard Bayesian environment where agents have private and strict preference orderings on a finite set of alternatives. Our main domains of preferences over alternatives are even larger than a broad class of domains — a few of its constituents being the unrestricted domain, the single-peaked domain, and the single-dipped domain. With independent and generic priors, we show that LOBIC of a mechanism combined with unanimity implies the tops-only property. Furthermore, we find a subclass of the domains where a mechanism with LOBIC and unanimity is dictatorial. We study the sufficiency of local incentive constraints for full incentive constraints and the relationship between LOBIC and dominant strategy incentive compatibility.

**Keywords:** Incentive compatibility, Local incentive compatibility, Tops-only property, Dictatorship, Connected domains, Unanimity

**JEL Classification:** C72, D01, D02, D72, D82.

## 1 Introduction

Incentive compatibility of a social choice function (or a voting mechanism) has been one of the foremost concerns in mechanism design. Gibbard (1973), Satterthwaite (1975), and Moulin (1980) are some seminal papers that address this subject. However, when the number of alternatives or the set of admissible preferences is large,

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it becomes onerous and costly to verify that a social choice function (scf) satisfies every incentive constraint. Therefore, the sufficiency of *local* constraints — on small distortions in reporting preferences — for full incentive compatibility has attracted substantial attention from the mechanism design literature: some recent papers include Carroll (2012), Sato (2013), and Mishra (2016). Particularly, we investigate the concept of local incentive compatibility suggested by Mishra (2016), locally ordinal Bayesian incentive compatibility.

Meanwhile, unanimity of an scf is a mild form of efficiency<sup>1</sup> — whenever every agent agrees on an alternative as the best, the scf should choose that alternative. Hence, it is quite natural to require an scf to be unanimous. Thus, we also mainly consider unanimous scfs.

This paper answers the following questions concerning locally ordinal Bayesian incentive compatible scfs combined with unanimity. First, when do those scfs invoke a strong and controversial property called *tops-onlyness*? Tops-onlyness exclusively responds to changes in the tops of preference profiles. For a mechanism designer, it is sufficient to consider scfs with this property as candidates. So, it can ease the complexity in voting mechanism design but prohibit free design. Second, when do the scfs have the undesirable property called *dictatorship*? Dictatorship requires the existence of a dictator whose top preference is always chosen. Third, when are the local incentive constraints sufficient to imply full incentive compatibility? In this case, the designer can extremely decrease the number of incentive constraints to be imposed upon agents. Finally, what is the relationship between local incentive compatibility and dominant strategy incentive compatibility (strategy proofness), which is extensively studied in the literature? These questions are not innovative but interesting since many other studies answer similar questions focusing on the dominant strategy incentive compatibility of unanimous scfs.

Our framework is built on a standard Bayesian environment where individuals have private and strict preference orderings on a finite set of alternatives. We consider profiles of the independent and *generic* priors introduced in Majumdar and Sen (2004) and studied in Mishra (2016)<sup>2</sup>. Additionally, we restrict our attention to deter-

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<sup>1</sup>Holströmes and Myerson (1983) classify the concepts of Pareto efficiency depending on the stage of information regarding the types of agents, among which, Ex-post Pareto efficiency is the weakest. Azrieli and Kim (2014) explain that any Ex-post Pareto efficient scf is unanimous.

<sup>2</sup>Specifically, Majumdar and Sen (2004) prove that their priors are generic in a topological sense under the unrestricted domain. Mishra (2016) explains that the identical proof works in restricted

ministic ordinal scfs that only account for the ordinal preferences of individuals. There are two main concepts of incentive compatibility for these scfs: dominant strategy incentive compatibility (DSIC) and ordinal Bayesian incentive compatibility (OBIC) introduced by d’Aspremont and Peleg (1988). An scf is OBIC if for any agent, the interim outcome probability vector from truth-telling first-order stochastically dominates any other interim outcome probability vector obtained from lying. Mishra (2016) defines OBIC with respect to generic priors (G-OBIC). Local incentive constraints are weakened versions of each full incentive compatibility (DSIC and G-OBIC); those that merely pertain to local distortions: local dominant strategy incentive compatibility (LDSIC)<sup>3</sup>, and generic-local ordinal Bayesian incentive compatibility (G-LOBIC).

For the domain of preferences, we assume that the set of admissible preferences is *connected* following the notions in Sato (2013): from any preference ordering to another, there is a path of *adjacent* orderings that are preference orderings only with the difference of a pair of consecutively ranked alternatives. That is, any *large* distortion in preferences can be decomposed into a sequence of *local (or small)* distortions. Additionally, Sato (2013) defines an important subclass of connected domains, *connected domains without restoration*, where a certain type of decomposition is possible. For any pair of preferences, there exists a path along which any adjacent distortion is not reversed (or restored) later in the sequence. Many well-known and widely studied domains, such as the unrestricted domain and the full single-peaked domain, lie in this class of domains. However, we consider an even larger class of domains, *weakly connected domains without restoration*. To be clear, Sato (2013) introduces these domains as necessary yet not sufficient domains for the equivalence between LDSIC and DSIC. Since this class of domains is sufficient for our main results, we name it. Whereas the former class necessitates the existence of a single path where the ranking between *any* two alternatives is not restored, our main domains merely demand the existence of one path for *each* pair of alternatives where the ranking between them is not restored. Note that the gap between two classes of domains can be larger as the number of alternatives grows.

We discuss our results referring to the existing literature. Our first main result shows that on the weakly connected domain without restoration, G-LOBIC scfs with unanimity are tops-only (Theorem 1). Tops-onlyness is desirable for a mechanism

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domains.

<sup>3</sup>LDSIC corresponds to AM-proof in Sato (2013).

designer (or a social planner) since it saves the cost of collecting and processing data. Tops-onlyness is also desirable for agents, who reveal their preferences in terms of privacy. However, tops-onlyness also restricts the design mechanism. Our result is in line with the literature on tops-onlyness — such as Weymark (2008) and Chatterji and Sen (2011) — which shows that DSIC with unanimity implies tops-onlyness on several domains. We generalize this result by weakening DSIC to G-LOBIC.

Second, we further find the subclass of weakly connected domain without restoration where G-LOBIC scfs with unanimity are dictatorial (Theorem 2). The property for the subclass is that for any adjacent preference orderings with the top swapped alternatives  $x, y$  in the domain, there exist two other preference orderings that show the disagreed preference over  $\{x, y\}$  and whose tops are not  $x, y$ . Note that these tops could be the same or not. We call this disagreement property of domain, which is in line with the approach of the literature on dictatorship — such as Aswal et al. (2003), Sato (2010), and Achuthankutty and Roy (2018) — which imposes the restriction on the top two alternatives of preference orderings. These authors study several domains where DSIC scf with unanimity are dictatorial. Another notable difference is that these studies necessitate the property of domain (called regular property in the literature) where every alternative should be the top of some preference orderings, which may be stronger as the number of alternatives grow. However, our property becomes relatively weaker.

Next, we study the sufficiency of local incentive constraints. Sato (2013) shows that weakly connected domains without restoration are necessary but not sufficient for the equivalence between LDSIC and DSIC. However, we show that assuming unanimity restores the equivalence between LDSIC and DSIC and invokes the equivalence of G-LOBIC and G-OBIC (Theorem 3). While the former equivalence has been discussed in the literature, to our knowledge, we are the first to study the equivalence of ordinal Bayesian incentive constraints. This result is particularly relevant when a mechanism designer considers DSIC to be demanding. For example, DSIC and unanimous scfs are inevitably dictatorial on the unrestricted domain leaving OBIC scfs as natural substitutes. In addition, we show that connected domains without restoration are sufficient but not necessary for the equivalence between G-LOBIC and G-OBIC without unanimity, which highlights the importance of unanimity (Proposition 1).

Finally, we illustrate one of the theoretical implications of our results. Mishra (2016) — the work most related to ours — studies the conditions for the equivalence of

G-LOBIC and DSIC in restricted domains. We strengthen Mishra’s (2016) results by generalizing the sufficient domain to the weakly connected domain without restoration and relaxing the exogenous restriction of tops-onlyness on scfs (Theorem 4)

The rest of the paper is organized as follows. We present a detailed framework in Section 2. In Section 3, we demonstrate our main results. All of the proofs are in the appendix.

## 2 The Model

### 2.1 Framework

Consider a standard Bayesian environment with private types.<sup>4</sup> The set of agents is  $N = \{1, 2, \dots, n\}$ , and the set of alternatives is  $A$  with  $m \equiv |A| \geq 3$ . Let  $\mathcal{P}$  denote the set of all strict linear orders over  $A$ . Then,  $\mathcal{P}$  is the *unrestricted domain*, and a proper subset  $\mathcal{D} \subset \mathcal{P}$  is a restricted domain. Each agent  $i \in N$  has a private preference ordering (or a type)  $P_i \in \mathcal{D}$ . For any preference ordering  $P \in \mathcal{D}$  and any pair of alternatives  $\{a, b\} \in A$ ,  $a P b$  if and only if  $a$  is strictly preferred to  $b$  by  $P$ .

A deterministic and ordinal scf is a mapping,  $f : \mathcal{D}^n \rightarrow A$ . We focus on scfs that choose an alternative whenever it is agreed by all agents as the best alternative; unanimous scfs. For any preference ordering  $P \in \mathcal{D}$  and any integer  $k \in K \equiv \{1, \dots, m\}$ , let  $P(k)$  denote the  $k$ th-ranked alternative for  $P$ .

**Definition 1.** An scf  $f$  is **unanimous** if for any  $\mathbf{P} \in \mathcal{D}^n$  and  $a \in A$ ,  $f(\mathbf{P}) = a$  whenever  $a = P_i(1)$  for every agent  $i \in N$ .

We assume that each agent independently draws their preference using a probability distribution  $\mu_i : \mathcal{D} \rightarrow [0, 1]$ , which is common knowledge for every agent. For any  $Q \subseteq \mathcal{D}^{n-1}$ , agent  $i$ ’s belief of others having a preference profile in  $Q$  is  $\mu(Q) = \sum_{\mathbf{P}_{-i} \in Q} \times \mu_j(P_j)$ . We mainly consider the following profile of priors.

**Definition 2** (Majumdar and Sen 2004). A profile of priors  $\{\mu_i\}_{i \in N}$  is **generic** if for every  $Q, R \subseteq \mathcal{D}^{n-1}$ , we have  $[\mu(Q) = \mu(R)] \Rightarrow [Q = R]$ .

The generic prior requires that the probabilities of any two distinct subsets of preference profiles of  $n - 1$  agents should be different. Majumdar and Sen (2004) and Mishra (2016) mathematically show that these priors are generic in a topological sense. Denote this set of distributions by  $C$  and the set of all independent distributions

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<sup>4</sup>We borrow several concepts and notations from Mishra (2016).

by  $\Delta^I$ . They say that  $C$  is large and generic in  $\Delta^I$  with two properties: (1)  $C$  is open and dense in  $\Delta^I$  and (2)  $\Delta^I - C$  has Lebesgue measure zero.<sup>5</sup>

## 2.2 Concepts of Incentive Compatibility

We now define several IC constraints: from the most stringent constraint, DSIC.

**Definition 3.** An scf is **dominant strategy incentive compatible (DSIC)** if for every  $i \in N$ , every  $P_i \in \mathcal{D}$ , and every  $\mathbf{P}_{-i} \in \mathcal{D}^{n-1}$ , there exists no  $P'_i \in \mathcal{D}$  such that  $f(P'_i, \mathbf{P}_{-i}) P_i f(P_i, \mathbf{P}_{-i})$ .

A weaker concept of incentive compatibility is LDSIC. We define some necessary concepts. For any two types  $P, P' \in \mathcal{D}$ , we say that  $\mathbf{P}'$  is an  **$(a, b)$ -swap of  $\mathbf{P}$**  if for some  $a, b \in A$  and  $k \in K$ ,  $P(k) = P'(k+1) = a$ ,  $P(k+1) = P'(k) = b$  and  $P'(j) = P(j)$  for all  $j \in K \setminus \{k, k+1\}$ . Additionally, a pair of types  $P, P' \in \mathcal{D}$  is **adjacent** if  $P'$  is an  $(a, b)$ -swap of  $P$  for some  $\{a, b\} \subset A$  and denote the adjacent alternatives by  $A(P, P') = \{a, b\}$ .

**Definition 4** (Mishra 2016). An scf is **locally dominant strategy incentive compatible (LDSIC)** if for every  $i \in N$ , every  $P_i \in \mathcal{D}$  and every  $\mathbf{P}_{-i} \in \mathcal{D}^{n-1}$ , there exists no **adjacent** type  $P'_i \in \mathcal{D}$  to  $P_i$  such that  $f(P'_i, \mathbf{P}_{-i}) P_i f(P_i, \mathbf{P}_{-i})$ .

Denote the union of  $a$  and the set of alternatives preferred to alternative  $a$  to type  $P_i$  as  $B(a, P_i) = \{a' \in A : a' = a \text{ or } a' P_i a\}$ . In addition, for each agent  $i \in N$ , let  $\pi_i^f(a, P_i) \equiv \sum_{\mathbf{P}_{-i} \in \mathcal{D}^{n-1} : f(P_i, \mathbf{P}_{-i}) = a} \mu(\mathbf{P}_{-i})$ . Now we define the ordinal notion of Bayesian incentive compatibility.

**Definition 5** (d'Aspremont and Peleg 1988). An scf  $f$  is **ordinally Bayesian incentive compatible (OBIC)** with respect to  $\{\mu_i\}_{i \in N}$  if for all  $i \in N$ ,  $P_i, P'_i \in \mathcal{D}$  and all  $a \in A$ , we have

$$\pi_i^f(B(a, P_i), P_i) \geq \pi_i^f(B(a, P_i), P'_i).^6$$

$f$  is **G-OBIC** if it is OBIC with respect to a profile of generic priors  $\mu$ .

<sup>5</sup>The important non-generic prior is the uniform prior that assigns the same probability to every preference.

<sup>6</sup>With some abuse of notation, let  $\pi_i^f(B(a, P_i), P'_i)$  denote the sum of probabilities of preference profiles such that  $f(\mathbf{P}) \in B(a, P_i)$  when reporting  $P'_i$ . That is,  $\pi_i^f(B(a, P_i), P'_i) \equiv \sum_{\mathbf{P}_{-i} \in \mathcal{D}^{n-1} : f(P'_i, \mathbf{P}_{-i}) \in B(a, P_i)} \prod_{j \neq i} \mu_j(P_j)$

As DSIC is weakened to LDSIC, OBIC can be weakened in the same spirit.

**Definition 6** (Mishra 2016). An scf  $f$  is **locally ordinally Bayesian incentive compatible (LOBIC)** with respect to  $\{\mu_i\}_{i \in N}$  if for all  $i \in N$ , for all  $a \in A$ , and for all pair of adjacent types  $P_i, P'_i \in D$ , we have

$$\pi_i^f(B(a, P_i), P_i) \geq \pi_i^f(B(a, P_i), P'_i). \quad (1)$$

$f$  is **G-LOBIC** if it is LOBIC with respect to a profile of generic priors  $\{\mu_i\}_{i \in N}$ .

### 2.3 Domains of Preferences

Finally, we define some concepts for the domain of preferences. Any pair of types  $P, P' \in \mathcal{D}$  is **connected** if there exists a sequence of types  $(P = P^0, P^1, \dots, P^h, P^{h+1} = P')$  in  $\mathcal{D}$  such that for every  $l \in \{0, 1, \dots, h\}$ ,  $P^l$  and  $P^{l+1}$  are adjacent.<sup>7</sup> For each pair  $\{a, b\} \subset A$ , a sequence in  $\mathcal{D}$  is **with  $\{a, b\}$ -restoration** if for some distinct  $l, l' \in \{0, 1, \dots, h\}$ ,  $A(P^l, P^{l+1}) = A(P^{l'}, P^{l'+1}) = \{a, b\}$  and **without  $\{a, b\}$ -restoration** if there exist no such swaps. A sequence is **without restoration** if it is without  $\{a, b\}$ -restoration for any  $\{a, b\} \subset A$ .

We first define a domain in which every pair of preference orderings are connected without restoration.

**Definition 7** (Sato 2013). A domain  $\mathcal{D} \subseteq \mathcal{P}$  is **connected without restoration** if any pair of types  $P, P' \in \mathcal{D}$  is connected without restoration.

Mishra (2016) discusses several examples of connected domains without restoration such as the unrestricted domain, the single-peaked domain, the single-dipped domain, and some single-crossing domains. However, we consider an even broader class of domains that includes connected domains without restoration: weakly connected domains without restoration.

**Definition 8.** A domain  $\mathcal{D} \subseteq \mathcal{P}$  is **weakly connected without restoration** if for each  $\{a, b\} \subset A$ , any pair of types  $P, P' \in \mathcal{D}$  is connected without  $\{a, b\}$ -restoration.

The difference between two domains is in the requirement of special sequence of adjacent types for any two pairs of types. The weakly connected domains without restoration require such a sequence for each pair of alternatives  $a, b$  while the connected domains without restoration require the single sequence for all pairs of

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<sup>7</sup>We often call such a sequence a path.

$P^1$	$P^2$	$P^3$	$P^4$	$P^5$	$P^6$	$P^7$	$P^8$	$P^9$	$P^{10}$
$x$	$y$	$y$	$y$	$y$	$y$	$z$	$z$	$z$	$x$
$y$	$x$	$x$	$w$	$w$	$z$	$y$	$y$	$x$	$z$
$z$	$z$	$w$	$x$	$z$	$w$	$w$	$x$	$y$	$y$
$w$	$w$	$z$	$z$	$x$	$x$	$x$	$w$	$w$	$w$

Table 1: Preference orderings in Example 1.

$$\begin{array}{c}
P^1 \underline{\{x, y\}} P^2 \underline{\{z, w\}} P^3 \underline{\{x, w\}} P^4 \underline{\{x, z\}} P^5 \\
\begin{array}{cc}
| \{y, z\} & \{z, w\} | \\
P^{10} \underline{\{x, z\}} P^9 \underline{\{x, y\}} P^8 \underline{\{x, w\}} P^7 \underline{\{y, z\}} P^6
\end{array}
\end{array}$$

Figure 1: Adjacency between preference orderings in Example 1.

alternatives. The natural question is the gap between the two domains. When the number of alternatives is three, there is no gap. However, if the number is more than three, we can easily observe the gap. The following example shows the gap for the case of four alternatives.

**Example 1.** Let  $N = \{1, 2\}$  and  $A = \{x, y, z, w\}$ . Consider the domain  $\mathcal{D}$  composed of preference orderings from  $P^1$  to  $P^{10}$  in Table 1.

Figure 1 shows the adjacency between preference orderings in Example 1. The notation  $P^1 \underline{\{x, y\}} P^2$  means that  $P^1$  and  $P^2$  are adjacent and  $A(P^1, P^2) = \{x, y\}$ . The other parts can be interpreted similarly. Let us check whether this domain is connected without restoration or not. The critical observation is that for  $P^1$  and  $P^6$ , all the two sequences connecting  $P^1$  and  $P^6$  are with  $\{z, w\}$  or  $\{y, z\}$ -restoration. However, for any pair of types in the domain, there exists a sequence without each pair of alternatives. Thus, it is not a connected domain without restoration, but a weakly connected domain without restoration.

Note that it is obvious that the greater the number of alternatives, the larger the gap between the two domains. As in the example, we do not require the regularity property for a domain where every alternative should be top of some preference ordering. This implies that the gap could be even larger than with it is the regularity

property.

### 3 Results

#### 3.1 Tops-onlyness

An scf is tops-only if it only takes into account the top alternative of each agent.

**Definition 9.** An scf  $f$  is **tops-only** if for any  $\mathbf{P}, \mathbf{P}' \in \mathcal{D}^n$ ,  $f(\mathbf{P}) = f(\mathbf{P}')$  whenever  $P_i(1) = P'_i(1)$  for all  $i \in N$ .

Our first main result shows that on the weakly connected domain without restoration, tops-onlyness is necessary for G-LOBIC scfs under unanimity.

**Theorem 1.** Let  $f : \mathcal{D}^n \rightarrow A$  be an scf where  $\mathcal{D} \subseteq \mathcal{P}$  is a weakly connected domain without restoration. If  $f$  is unanimous and G-LOBIC, it is tops-only.

#### 3.2 Dictatorship

An scf is dictatorial if there exists an agent (dictator) whose top alternative is chosen for any preference profile.

**Definition 10.** An scf  $f$  is **dictatorial** if there exists an agent  $i \in N$  such that for any  $\mathbf{P} \in \mathcal{D}^n$ ,  $f(\mathbf{P}) = P_i(1)$ .

We define another domain property, disagreement property that shows the disagreement of preference over the top-swapped alternatives.

**Definition 11.** A domain  $\mathcal{D} \subseteq \mathcal{P}$  satisfies a disagreement property if for any adjacent types  $P, P'$  such that  $P(1) = x$ ,  $P(2) = y$  and  $A(P, P') = \{x, y\}$ , there exists a pair of types  $\hat{P}, \bar{P}$  such that  $\hat{P}(1), \bar{P}(1) \notin \{x, y\}$  and  $x\hat{P}y, y\bar{P}x$ .

**Remark.**  $\hat{P}(1)$  and  $\bar{P}(1)$  could be the same or not. We observe that the domain in Example 1 satisfies the disagreement property with  $\hat{P}(1) = \bar{P}(1)$  for each pair of top-swapped alternatives,  $\{x, y\}, \{y, z\}, \{z, x\}$ . Additionally, note that this property concerns only the pair of top-swapped alternatives, not all alternatives.

Our second main result is that if a weakly connected domain without restoration satisfies the disagreement property, any G-LOBIC scf under unanimity is dictatorial.

**Theorem 2.** Let  $f : \mathcal{D}^n \rightarrow A$  be an scf where  $\mathcal{D} \subseteq \mathcal{P}$  is a weakly connected domain without restoration and satisfies a disagreement property. If  $f$  is unanimous and G-LOBIC, it is dictatorial.

The notable connected domain without restoration that does not satisfy the disagreement property is the single-peaked preference domain.

### 3.3 Local Domains and Sufficiency of Local Incentive Constraints

Mishra (2016) calls a domain *local* if LDSIC is equivalent to DSIC in that domain. Sato (2013) shows that connected domains without restoration are local. We show in the following proposition that the equivalence of G-LOBIC and G-OBIC also holds in these domains.

**Proposition 1.** Let  $f : \mathcal{D}^n \rightarrow A$  be an scf where  $\mathcal{D}$  is a connected domain without restoration. Then,  $f$  is G-LOBIC if and only if it is G-OBIC.

While G-LOBIC and LDSIC are sufficient, respectively, for G-OBIC and DSIC in connected domains without restoration, it can be shown that there exist scfs that are LDSIC but not DSIC nor OBIC in weakly connected domains without restoration<sup>8</sup>. However, the following theorem shows that under unanimity, the equivalences are restored.

**Theorem 3.** Let  $f : \mathcal{D}^n \rightarrow A$  be a unanimous scf where  $\mathcal{D}$  is a weakly connected domain without restoration. Then,  $f$  is G-LOBIC (resp. LDSIC) if and only if it is G-OBIC (resp. DSIC).

### 3.4 Locally Ordinal Bayesian Incentive Compatibility and Dominant Strategy Incentive Compatibility

Using the results from above, we study the relationship between G-LOBIC and DSIC. Mishra (2016) investigates two weak versions of Maskin monotonicity in the context of this relationship.

**Definition 12** (Mishra 2016). An scf  $f$  satisfies **elementary monotonicity** if for every  $i \in N$ , every  $\mathbf{P}_{-i} \in \mathcal{D}^{n-1}$ , and every  $P_i, P'_i \in \mathcal{D}$  such that  $P'_i$  is an  $(a, b)$ -swap of  $P_i$  for some  $a, b \in A$  and  $f(P_i, \mathbf{P}_{-i}) = b$ , we have  $f(P'_i, \mathbf{P}_{-i}) = b$ .

The author first shows that G-LOBIC combined with elementary monotonicity implies DSIC in local domains. Then, the author further relaxes monotonicity to hold only in a restricted set of preference profiles. For a domain  $\mathcal{D} \subset \mathcal{P}$ , a profile of

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<sup>8</sup>Example 3.2 in Sato (2013) serves this purpose.

preferences  $\mathbf{P} \in \mathcal{D}^n$  is a **top-2 profile** if for every  $i, j \in N$ ,  $P_i(k) = P_j(k)$  for all  $k > 2$ . Let  $\mathcal{D}^2(2)$  be the set of all top-2 profiles in  $\mathcal{D}$ .

**Definition 13** (Mishra 2016). An scf  $f : \mathcal{D}^n \rightarrow A$  satisfies **weak elementary monotonicity** if  $f$  restricted to  $\mathcal{D}^n(2)$  satisfies elementary monotonicity.

Similarly to our spirit, Mishra (2016) proves that under unanimity, elementary monotonicity can be replaced by weak elementary monotonicity for the sufficiency of G-LOBIC for DSIC in the single-peaked domain. In connected domains without restoration, tops-onlyness is additionally necessary for the sufficiency. However, the following lemma shows that an intermediate step (for the sufficiency of G-LOBIC for LDSIC) works without the tops-only property even in weakly connected domains without restoration.

**Lemma 1.** Let  $f : \mathcal{D}^n \rightarrow A$  be a unanimous scf where  $\mathcal{D}$  is a weakly connected domain without restoration. Then,  $f$  is LDSIC if and only if it is G-LOBIC and satisfies weak elementary monotonicity.

Combining Theorem 3 and Lemma 1 leads to our last theorem.

**Theorem 4.** Let  $f : \mathcal{D}^n \rightarrow A$  be a unanimous scf where  $\mathcal{D}$  is a weakly connected domain without restoration. Then,  $f$  is DSIC if and only if it is G-LOBIC and satisfies weak elementary monotonicity.

Theorem 4 is a twofold strengthening of Mishra’s (2016) results. First, whereas Mishra (2016) presents the sufficient conditions contingent on the types of domains — single-peaked or connected without restoration —, we present an inclusive result. That is, we show that tops-onlyness required in Mishra’s (2016) result is redundant. In fact, this redundancy follows from Theorem 1. Second, we extend the domain. Showing the equivalence of LDSIC with DSIC is a crucial step in the proof of Theorem 4. While Mishra (2016) relies on the observation that connected domains without restoration are local, we use Theorem 3 for this step. Recall that the class of weakly connected domains without restoration is larger than the class of local domains in Mishra (2016).

## Appendix

**Proof of Theorem 1.** Suppose for the sake of contradiction that  $f$  is unanimous and G-LOBIC, but not tops-only. Then, there exists an agent (for example, agent

1),  $\mathbf{P}_{-1} \in \mathcal{D}^{n-1}$  and two types  $P_1, \bar{P}_1 \in \mathcal{D}$  such that  $a^* \equiv P_1(1) = \bar{P}_1(1)$  and  $f(\bar{P}_1, \mathbf{P}_{-1}) \neq f(P_1, \mathbf{P}_{-1})$ .<sup>9</sup>

The first step of our proof is to show that the outcome of  $f$  is invariant to adjacent manipulations in preferences when the top alternative is fixed (Lemma 4). To do so, we introduce a property called swap monotonicity (SM) and a result — both from Mishra (2016) — that any G-LOBIC scf satisfies SM (Lemma 2)<sup>10</sup>. We show that on weakly connected domains without restoration, an scf that satisfies SM has a special property (Lemma 3) and use Lemma 2 and Lemma 3 to prove Lemma 4. Finally, we generalize the invariance of the outcome of  $f$  for larger manipulations in preferences with common tops, which shows that  $f$  is tops-only.

**Definition 14** (Mishra 2016). An scf  $f$  satisfies **swap monotonicity (SM)** if for every  $i \in N$  and  $P_i, P'_i \in \mathcal{D}$  such that  $A(P_i, P'_i) = \{a, b\} \subset A$ , we have for every  $\mathbf{P}_{-i} \in \mathcal{D}^{n-1}$  that  $f(P'_i, \mathbf{P}_{-i}) = f(P_i, \mathbf{P}_{-i})$  if  $f(P_i, \mathbf{P}_{-i}) \notin \{a, b\}$  and  $f(P'_i, \mathbf{P}_{-i}) \in \{a, b\}$  if  $f(P_i, \mathbf{P}_{-i}) \in \{a, b\}$ .

**Lemma 2** (Mishra 2016). Let  $f : \mathcal{D}^n \rightarrow A$  be a G-LOBIC scf where  $\mathcal{D} \subseteq \mathcal{P}$ . Then,  $f$  satisfies swap monotonicity.

**Lemma 3.** Let  $f : \mathcal{D}^n \rightarrow A$  be an scf where  $\mathcal{D}$  is a weakly connected domain without restoration. If  $f$  satisfies swap monotonicity, then it has the following property. For any  $i \in N$ ,  $P_i \in \mathcal{D}$ ,  $\mathbf{P}_{-i} \in \mathcal{D}^{n-1}$  and  $\{a, b\} \subset A$ , if  $f(P_i, \mathbf{P}_{-i}) \neq f(P'_i, \mathbf{P}_{-i})$  where  $P'_i$  is an  $(a, b)$ -swap of  $P_i$ , then  $f(P_i, \mathbf{P}_{-i}) = f(P_i, \mathbf{P}'_{-i})$  and  $f(P'_i, \mathbf{P}'_{-i}) = f(P'_i, \mathbf{P}_{-i})$  for any  $\mathbf{P}'_{-i} \in \mathcal{D}^{n-1}$  such that for every  $j \in N \setminus \{i\}$ ,  $a P'_j b$  if and only if  $a P_j b$ .

*Proof.* Without loss of generality (WLOG), assume that  $f(P_1, \mathbf{P}_{-1}) \neq f(P'_1, \mathbf{P}_{-1})$  for some  $P_1 \in \mathcal{D}$ ,  $\mathbf{P}_{-1} \in \mathcal{D}^{n-1}$  and  $\{a, b\} \subset A$  such that  $A(P_1, P'_1) = \{a, b\}$ . Then by SM,  $\{f(P_1, \mathbf{P}_{-1}), f(P'_1, \mathbf{P}_{-1})\} = \{a, b\}$ . Assume without loss of generality that  $f(P_1, \mathbf{P}_{-1}) = a$  and  $f(P'_1, \mathbf{P}_{-1}) = b$  and that  $a P_2 b$ . Suppose  $P'_2 \in \mathcal{D}$  is a distinct type of agent 2 such that  $a P'_2 b$ . Then, there exists a path from  $P_2$  to  $P'_2$  ( $P_2 = P_2^0, P_2^1, \dots, P_2^h, P_2^{h+1} = P'_2$ ) in  $\mathcal{D}$  that is without  $\{a, b\}$ -restoration. For simplicity, for any  $l \in \{0, \dots, h+1\}$ , let  $\bar{f}(P_2^l) \equiv f(P_1, P_2^l, \mathbf{P}_{-\{1,2\}})$  and  $\bar{f}'(P_2^l) \equiv f(P'_1, P_2^l, \mathbf{P}_{-\{1,2\}})$ .

We argue that  $\forall l \in \{0, \dots, h\}$ ,  $[\bar{f}(P_2^l) = a, \text{ and } \bar{f}'(P_2^l) = b] \Rightarrow [\bar{f}(P_2^{l+1}) = a \text{ and } \bar{f}'(P_2^{l+1}) = b]$ . Since the path is without  $\{a, b\}$ -restoration,  $A(P_2^l, P_2^{l+1}) \neq \{a, b\}$ .

<sup>9</sup> $\mathbf{P}_{-1} \equiv (P_2, \dots, P_n)$ ,  $\mathbf{P}_{-\{1,2\}} \equiv (P_3, \dots, P_n)$ .

<sup>10</sup>This result is conveniently used in the proofs of Mishra's (2016) results and also has a crucial role in ours.

This leaves two possible cases:  $A(P_2^l, P_2^{l+1}) \cap \{a, b\} = \emptyset$  or  $A(P_2^l, P_2^{l+1}) \cap \{a, b\} \neq \emptyset$ . If it is the former case, then by SM,  $\bar{f}(P_2^{l+1}) = a$  and  $\bar{f}'(P_2^{l+1}) = b$ . For the latter case, assume without loss of generality that  $A(P_2^l, P_2^{l+1}) = \{a, c\}$  where  $c \in A$  and  $c \neq b$ . Then,  $\bar{f}(P_2^{l+1}) \in \{a, c\}$  and  $\bar{f}'(P_2^{l+1}) = b$  by SM. However, if  $\bar{f}(P_2^{l+1}) = c$ , then  $\bar{f}'(P_2^{l+1}) = c$  since  $A(P_1, P_1') = \{a, b\}$ , which is a contradiction. Therefore,  $\bar{f}(P_2^{l+1}) = a$  and  $\bar{f}'(P_2^{l+1}) = b$ .

Since  $\bar{f}(P_2) = a$  and  $\bar{f}'(P_2) = b$ ,  $\bar{f}(P_2') = a$  and  $\bar{f}'(P_2') = b$ . Finally, we can apply the same process to agent 3, 4, ..., and  $n$  to complete the proof.  $\square$

**Lemma 4.** Let  $f : \mathcal{D}^n \rightarrow A$  be a unanimous scf where  $\mathcal{D}$  is a weakly connected domain without restoration. If  $f$  is G-LOBIC, then for every agent  $i \in N$ , their type  $P_i$  and  $\mathbf{P}_{-i} \in \mathcal{D}^{n-1}$ ,  $f(P_i', \mathbf{P}_{-i}) = f(P_i, \mathbf{P}_{-i})$  for any adjacent type  $P_i'$  of  $P_i$  such that  $P_i'(1) = P_i(1)$ .

*Proof.* Suppose on the contrary that for some agent (for example, agent 1),  $f(P_1', \mathbf{P}_{-1}) \neq f(P_1, \mathbf{P}_{-1})$  for some adjacent preferences  $P_1, P_1'$  with  $a^* \equiv P_1(1) = P_1'(1)$  and some  $\mathbf{P}_{-1} \in \mathcal{D}^{n-1}$ . Let  $A(P_1, P_1') = \{a, b\}$  for some  $a, b \in A$  and  $a \neq b$ . Then by SM, either  $[f(P_1, \mathbf{P}_{-1}) = a \text{ and } f(P_1', \mathbf{P}_{-1}) = b]$  or  $[f(P_1, \mathbf{P}_{-1}) = b \text{ and } f(P_1', \mathbf{P}_{-1}) = a]$  holds. Without loss of generality, assume the former case and let  $a \equiv P_1(k)$  and  $b \equiv P_1(k+1)$ . Then, we can modify the preference orderings of other agents contingent on their relative rankings between  $a$ ,  $b$ , and  $a^*$ . For simplicity, for any  $l \in \{0, \dots, h+1\}$ , let  $\bar{f}(P_2^l) \equiv f(P_1, P_2^l, \mathbf{P}_{-\{1,2\}})$  and  $\bar{f}'(P_2^l) \equiv f(P_1', P_2^l, \mathbf{P}_{-\{1,2\}})$ .

Case 1: If  $P_2(1) = a^*$ , take  $P_2' = P_2$ . Then  $\bar{f}(P_2') = a$  and  $\bar{f}'(P_2') = b$ .

Case 2: If  $P_2(1) \neq a^*$  and  $a P_2 b$ , take  $P_2' = P_1$  so that the ranking between  $a$  and  $b$  for  $P_2'$  matches that of  $P_2$ . Then, by Lemma 3,  $\bar{f}(P_2') = a$  and  $\bar{f}'(P_2') = b$ .

Case 3: Analogously, if  $P_2(1) \neq a^*$  and  $b P_2 a$ , take  $P_2' = P_1'$ . Then,  $\bar{f}(P_2') = a$  and  $\bar{f}'(P_2') = b$ .

We can apply the same procedure to the preference orderings of agent 3, 4, ..., and  $n$  to get  $f(P_1, P_2', P_3', \dots, P_{n-1}', P_n') = a \neq a^*$  and  $f(P_1', P_2', P_3', \dots, P_{n-1}', P_n') = b \neq a^*$  where  $P_j(1) = P_j'(1) = a^*$  for all  $j \in N$ , which contradicts the unanimity of  $f$ .  $\square$

Now, we use the lemmas above to show that  $f(\bar{P}_1, \mathbf{P}_{-1}) = f(P_1, \mathbf{P}_{-1})$ , which completes the proof of Theorem 1. It suffices to show that the equality holds if  $f(P_1, \mathbf{P}_{-1}) \neq a^*$  since, by this fact,  $f(P_1, \mathbf{P}_1) = a^* \neq f(\bar{P}_1, \mathbf{P}_{-1})$  is not possible.

Suppose  $f(P_1, \mathbf{P}_{-1}) \neq a^*$ . Since  $\mathcal{D}$  is weakly connected without restoration, there exists a path connecting  $P_1$  and  $\bar{P}_1$  ( $P_1 = P_1^0, P_1^1, \dots, P_1^h, P_1^{h+1} = \bar{P}_1$ ) that is without

$\{a^*, f(P_1, \mathbf{P}_{-1})\}$ -restoration. If  $P_1^l(1) = a^*$  for every  $1 \leq l \leq h$ , that is, alternative  $a^*$  is never swapped in the sequence, we can apply Lemma 4 to every swap of the sequence so that  $f(\bar{P}_i, \mathbf{P}_{-1}) = f(P_1, \mathbf{P}_{-1})$ . On the other hand, it is possible that there exists  $1 \leq l' \leq h$  such that  $P_1^{l'}(1) \neq a^*$ . Since the path is without  $\{a^*, f(P_1, \mathbf{P}_{-1})\}$ -restoration, we have  $a^* P_1^l f(P_1, \mathbf{P}_{-1})$  for every  $1 \leq l \leq h$ . However, Lemma 2 and Lemma 4 imply that for any  $0 \leq l \leq h$ ,  $f(P_1^l, \mathbf{P}_{-1}) \neq f(P_1^{l+1}, \mathbf{P}_{-1})$  is only possible when  $A(P_1^l, P_1^{l+1}) = \{P_1^l(1), P_1^{l+1}(1)\}$  and  $f(P_1^l, \mathbf{P}_{-1}) \in A(P_1^l, P_1^{l+1})$ . Therefore,  $f(P_1^{l+1}, \mathbf{P}_{-1}) = f(P_1^l, \mathbf{P}_{-1})$  for every  $1 \leq l \leq h$  and, thus,  $f(\bar{P}_1, \mathbf{P}_{-1}) = f(P_1, \mathbf{P}_{-1})$ .

Finally, suppose  $f(P_1, \mathbf{P}_{-1}) = a^*$  and assume that  $f(\bar{P}_1, \mathbf{P}_{-1}) \neq a^*$  for the sake of contradiction. Then, we can apply the argument above to  $\bar{P}_1$ , which results in  $f(P_1, \mathbf{P}_{-1}) = f(\bar{P}_1, \mathbf{P}_{-1}) \neq a^*$ . □

### ***Proof of Theorem 2.***

Step 1) On the assumed domain, we claim that there exists a pair of alternatives  $x, y$  and adjacent types  $P, P'$  such that  $P(1) = x$ ,  $P(2) = y$  with a pair of types  $\hat{P}, \bar{P}$  such that  $\hat{P}(1) = \bar{P}(1) \notin \{x, y\}$  and  $x\hat{P}y, y\bar{P}x$ . To prove the claim, assume that for every such pair of alternatives  $x, y$ , there is no pair of types  $\hat{P}, \bar{P}$  such that  $\hat{P}(1) = \bar{P}(1) \notin \{x, y\}$  and  $x\hat{P}y, y\bar{P}x$ . Then, consider the path from  $P$  to a type  $\hat{P}$  with  $P(1) \neq \hat{P}(1)$  and  $x\hat{P}y$  without  $\{x, y\}$ -restoration and, similarly, a path from  $\hat{P}$  to a type  $\tilde{P}$  with  $\hat{P}(1) \neq \tilde{P}(1)$  and  $x\tilde{P}y$  without  $\{x, y\}$ -restoration,..., until the last different top alternative. For example, the last top swapped alternatives,  $\{l, m\}$  with  $\underline{P}, \underline{P}'$  such that  $\underline{P}(1) = l$ ,  $\underline{P}(2) = m$ . On the other hand, we can also construct a long path from  $P'$  to the last type with the different top and the same preference over  $\{x, y\}$  with  $P'$ . Note that all top alternatives in the domain are divided into two groups according to the preference over  $\{x, y\}$  of types. Then, consider the whole path connected with the above two long paths. The path contains types with all different tops and only one top connection with  $\{l, m\}$ . Additionally, since the domain is connected and there is no pair of types  $\hat{P}, \bar{P}$  such that  $\hat{P}(1) = \bar{P}(1) \notin \{l, m\}$  and  $l\hat{P}m, m\bar{P}l$ , all types on the path should show the same preference over  $\{l, m\}$  with  $\underline{P}$ . However, the disagreed domain requires that we have another type with the reversal preference over  $\{l, m\}$  outside the path. This type should share one top alternative with a type on the path, which leads to a contradiction in the assumption in the claim.

Step 2) From the above claim, we can start adjacent types  $P, P'$  such that  $P(1) =$

$x$ ,  $P(2) = y$ , and  $A(P, P') = \{x, y\}$  with a pair of types  $\hat{P}, \bar{P}$  such that  $\hat{P}(1) = \bar{P}(1) \notin \{x, y\}$  and  $x\hat{P}y, y\bar{P}x$ . By unanimity,  $f(P, P, \dots, P) = x$  and  $f(P', P', \dots, P') = y$ . Consider the change of type for each agent from  $P$  to  $P'$ , and there exists an agent  $i$  such that  $f(P_i = P, \mathbf{P}_{-i}) = x$  and  $f(P_i = P', \mathbf{P}_{-i}) = y$ . For such a  $\mathbf{P}_{-i} = (P_j, \mathbf{P}_{-\{i, j\}})$ , there are two cases where  $xP_jy$  or  $yP_jx$ . Consider  $xP_jy$  and the other case is similar. Recall, there exists a pair of types  $\hat{P}_j, \bar{P}_j$  such that  $\hat{P}_j(1) = \bar{P}_j(1) \notin \{x, y\}$  and  $x\hat{P}_jy, y\bar{P}_jx$ . By Lemma 3,  $f(P_i = P, \hat{P}_j, \mathbf{P}_{-\{i, j\}}) = x$  and  $f(P_i = P', \hat{P}_j, \mathbf{P}_{-\{i, j\}}) = y$ . By Theorem 1,  $f$  is tops only, which implies that  $f(P_i = P, \bar{P}_j, \mathbf{P}_{-\{i, j\}}) = x$  and  $f(P_i = P', \bar{P}_j, \mathbf{P}_{-\{i, j\}}) = y$ . By Lemma 3, for any type  $P_j$  such that  $xP_jy$  or  $yP_jx$ , we have that  $f(P_i = P, P_j, \mathbf{P}_{-\{i, j\}}) = x$  and  $f(P_i = P', P_j, \mathbf{P}_{-\{i, j\}}) = y$ . Note that this argument holds for any agent  $j \neq i$ . Thus,  $f(P_i(1) = x, \mathbf{P}_{-i}) = x$  and  $f(P_i(1) = y, \mathbf{P}_{-i}) = y$  for any  $\mathbf{P}_{-i}$ .

Step 3) For other adjacent types  $\tilde{P}, \tilde{P}'$  such that  $\tilde{P}(1) = z, \tilde{P}(2) = k$ , we first claim that  $f(P_i = \tilde{P}, \mathbf{P}_{-i}), f(P_i = \tilde{P}', \mathbf{P}_{-i}) \in \{z, k\}$ . Similar to Step 1), the unanimity requires that  $f(\tilde{P}, \tilde{P}, \dots, \tilde{P}) = z$  and  $f(\tilde{P}', \tilde{P}', \dots, \tilde{P}') = k$ , and there exists an agent  $i'$  and  $\mathbf{P}_{-i'}$  such that  $f(P_{i'} = \tilde{P}, \mathbf{P}_{-i'}) = z$  and  $f(P_{i'} = \tilde{P}', \mathbf{P}_{-i'}) = k$ . This agent  $i'$  should be the agent  $i$  in step 1). Otherwise, Lemma 3 and SM require that  $f(P_{i'} = \tilde{P}, P_i = P, \mathbf{P}_{-\{i', i\}}), f(P_{i'} = \tilde{P}', P_i = P, \mathbf{P}_{-\{i', i\}}) \in \{z, k\}$ , which leads to a contradiction to Step 2). Additionally, SM and Lemma 3 imply that  $f(P_i = \tilde{P}, P_j = \tilde{P} \text{ or } \tilde{P}', \mathbf{P}_{-\{i, j\}}), f(P_i = \tilde{P}', P_j = \tilde{P} \text{ or } \tilde{P}', \mathbf{P}_{-\{i, j\}}) \in \{z, k\}$  for any  $\mathbf{P}_{-\{i, j\}}$ , which proves the claim. WLOG, consider  $f(P_i = \tilde{P}, P_j = \tilde{P}, \mathbf{P}_{-\{i, j\}}) = z$  and  $f(P_i = \tilde{P}', P_j = \tilde{P}, \mathbf{P}_{-\{i, j\}}) = k$ . From the similar argument in Step 2), it is sufficient to show that  $f(P_i = \tilde{P}, P_j = \tilde{P}', \mathbf{P}_{-\{i, j\}}) = z$  and  $f(P_i = \tilde{P}', P_j = \tilde{P}', \mathbf{P}_{-\{i, j\}}) = k$  to prove the theorem. If not, SM requires that  $f(P_i = \tilde{P}, P_j = \tilde{P}', \mathbf{P}_{-\{i, j\}}) = z$  and  $f(P_i = \tilde{P}', P_j = \tilde{P}', \mathbf{P}_{-\{i, j\}}) = z$ . Since the domain satisfies a disagreement property, consider the type  $\check{P}$  such that  $\check{P}(1) \notin \{z, k\}$  and  $k\check{P}z$  and the path from  $\tilde{P}'$  to  $\check{P}$  without  $\{z, k\}$ -restoration. Then,  $f(P_i = \check{P}, P_j = \tilde{P}', \mathbf{P}_{-\{i, j\}}) = z$ , which contradicts the above claim.  $\square$

### **Proof of Proposition 1.**

We show that for any profile of generic priors  $\{\mu_i\}_{i \in N}$ ,  $f$  is LOBIC with respect to  $\{\mu_i\}_{i \in N}$  if and only if it is OBIC with respect to  $\{\mu_i\}_{i \in N}$ .

It suffices to show that LOBIC implies OBIC. Suppose that  $f$  is LOBIC with respect to  $\{\mu_i\}_{i \in N}$  and fix  $i \in N$ ,  $P_i, P'_i \in \mathcal{D}$ ,  $a \in A$ . Since  $\mathcal{D}$  is connected without restoration, there exists a path  $(P_i = P_i^0, P_i^1, \dots, P_i^h, P_i^{h+1} = P'_i)$  connecting  $P_i$  and  $P'_i$  without restoration. We show that for any  $l \in \{0, \dots, h\}$ , if  $\pi_i^f(B(a, P_i), P_i) \geq \pi_i^f(B(a, P_i), P_i^l)$ , then  $\pi_i^f(B(a, P_i), P_i) \geq \pi_i^f(B(a, P_i), P_i^{l+1})$ . There are two possible cases depending on the intersection of  $A(P_i^l, P_i^{l+1})$  and  $B(a, P_i)$ .

Case I: Suppose  $A(P_i^l, P_i^{l+1}) \cap B(a, P_i) = \emptyset$  or  $A(P_i^l, P_i^{l+1})$ . Then,  $\pi_i^f(B(a, P_i), P_i^l) = \pi_i^f(B(a, P_i), P_i^{l+1})$  by SM.

Case II: Suppose  $A(P_i^l, P_i^{l+1}) = \{x, y\} \subset A$  and  $\{x, y\} \cap B(a, P_i) = \{x\}$ . Since the path is without  $\{x, y\}$ -restoration,  $x P_i^l y$ . Furthermore, since  $f$  is LOBIC with respect to  $\{\mu_i\}_{i \in N}$ ,  $\pi_i^f(B(x, P_i^l), P_i^l) \geq \pi_i^f(B(x, P_i^l), P_i^{l+1})$  holds, which implies that  $\pi_i^f(x, P_i^l) \geq \pi_i^f(x, P_i^{l+1})$ . Moreover,  $\pi_i^f(b, P_i^l) = \pi_i^f(b, P_i^{l+1})$  holds for all  $b \in B(a, P_i) \setminus \{x\}$  by SM. Therefore,  $\pi_i^f(B(a, P_i), P_i^l) \geq \pi_i^f(B(a, P_i), P_i^{l+1})$ .

Since  $\pi_i^f(B(a, P_i), P_i) \geq \pi_i^f(B(a, P_i), P_i^1)$  holds by LOBIC, we have  $\pi_i^f(B(a, P_i), P_i) \geq \pi_i^f(B(a, P_i), P_i')$  by induction. □

**Proof of Theorem 3.** For the equivalence between G-LOBIC and G-OBIC, it suffices to show that G-LOBIC implies G-OBIC. Suppose  $f$  is G-LOBIC with respect to  $\{\mu_i\}_{i \in N}$ . For a pair of preferences  $P, P' \in \mathcal{D}$ , consider a path  $(P = P^0, P^1, \dots, P^h, P^{h+1} = P')$  connecting  $P$  and  $P'$ . For an agent  $i \in N$  and any alternative  $a \in A$ ,  $\pi_i^f(B(a, P), P) \geq \pi_i^f(B(a, P), P^1)$  holds by LOBIC of  $f$  with respect to  $\{\mu_i\}_{i \in N}$ . Proving that the following holds with any integer  $k \in \{1, \dots, h\}$  completes the proof:

$$\pi_i^f(B(a, P), P) \geq \pi_i^f(B(a, P), P^k) \Rightarrow \pi_i^f(B(a, P), P) \geq \pi_i^f(B(a, P), P^{k+1}). \quad (2)$$

Suppose (2) holds for some  $l$  ( $1 \leq l \leq h$ ). If  $P^{l+1}(1) = P^l(1)$ , then  $\pi_i^f(B(a, P), P^{l+1}) = \pi_i^f(B(a, P), P^l)$  since  $f$  is tops-only by Theorem 1. If  $P^{l+1}(1) \neq P^l(1)$ , we need only consider the case where  $P^l(1) \notin B(a, P)$  and  $P^{l+1}(1) \in B(a, P)$  since if otherwise,  $\pi_i^f(B(a, P), P^{l+1}) \leq \pi_i^f(B(a, P), P^l)$  by SM of  $f$  (as in Proposition 1).

Let  $x \equiv P^{l+1}(1)$ . Since  $\mathcal{D}$  is weakly connected without restoration, for any  $y \notin B(a, P)$ , there exists a path  $(P = \bar{P}^0, \bar{P}^1, \dots, \bar{P}^{\bar{h}}, \bar{P}^{\bar{h}+1} = P^{l+1})$  from  $P$  to  $P^{l+1}$  without  $\{x, y\}$ -restoration. Furthermore, since  $x P y$  and  $x P^{l+1} y$ , there does not exist  $m$  ( $1 \leq m \leq \bar{h}$ ) such that  $\bar{P}^m(1) = y$ . Then, tops-onlyness and SM of  $f$  imply that  $\pi_i^f(y, P) = \pi_i^f(y, \bar{P}^1) = \dots = \pi_i^f(y, \bar{P}^{\bar{h}}) = \pi_i^f(y, P^{l+1})$ . Therefore,  $\pi_i^f(A \setminus B(a, P), P^{l+1}) = \pi_i^f(A \setminus$

$B(a, P), P)$  and, thus,  $\pi_i^f(B(a, P), P^{l+1}) = \pi_i^f(B(a, P), P)$ .

For the equivalence between LDSIC and DSIC, it suffices to show that LDSIC implies DSIC. Suppose, on the contrary, that there exists an agent (for example, agent 1),  $P_1, P'_1 \in \mathcal{D}$ , and  $\mathbf{P}_{-1} \in \mathcal{D}^{n-1}$  such that  $f(P'_1, \mathbf{P}_{-1}) P_1 f(P_1, \mathbf{P}_{-1})$ . For simplicity, let  $x \equiv f(P'_1, \mathbf{P}_{-1})$  and  $y \equiv f(P_1, \mathbf{P}_{-1})$ . Then, if  $x P'_1 y$ , every path connecting  $P_1$  and  $P'_1$  is with  $\{x, y\}$ -restoration since tops-onlyness and LDSIC of  $f$  imply that for some preference  $\hat{P}_1$  in the path,  $\hat{P}_1(1) = y$ . This contradicts the assumption that  $\mathcal{D}$  is weakly connected without restoration. Therefore, we can assume that  $y P'_1 x$ . Consider a path  $(P_1 = P_1^0, P_1^1, \dots, P_1^h, P_1^{h+1} = P'_1)$  from  $P_1$  to  $P'_1$  without  $\{x, y\}$ -restoration, which exists since  $\mathcal{D}$  is weakly connected without restoration. Then, there exists  $l$  ( $0 \leq l \leq h$ ) such that  $A(P_1^l, P_1^{l+1}) = \{x, y\}$ ,  $P_1^l(1) = x$ ,  $f(P_1^l, \mathbf{P}_{-1}) = y$ , and  $f(P_1^{l+1}, \mathbf{P}_{-1}) = x$ . Since  $x P'_1 y$ , this contradicts LDSIC of  $f$ .

□

### ***Proof of Lemma 1.***

We first argue that if  $f$  is G-LOBIC and satisfies weak elementary monotonicity, it satisfies elementary monotonicity. Suppose not, that is, for an agent (agent 1), some  $\mathbf{P}_{-1} \in \mathcal{D}^{n-1}$  and  $P_1, P'_1 \in \mathcal{D}$ ,  $P'_1$  is a  $(a, b)$ -swap of  $P_1$  for some  $a, b \in A$ ,  $f(P_1, \mathbf{P}_{-1}) \neq a$  and  $f(P'_1, \mathbf{P}_{-1}) = a$ .

Since  $f$  is tops-only by Theorem 1,  $P_1(1) = a$ , and  $P_1(2) = b$ . Additionally,  $f(P_1, \mathbf{P}_{-1}) \in \{a, b\}$  by SM so  $f(P_1, \mathbf{P}_{-1}) = b$ . Now, we modify the preference orderings for all  $j \in N$  such that  $P_j \neq P_1$  and  $P_j \neq P'_1$ . Denote the set of such agents by  $N' \subset N$ .

If  $2 \notin N'$ , then, take  $P'_2 = P_2$ . Otherwise, take  $P'_2 = P_1$  if  $a P_2 b$  and  $P'_2 = P'_1$  if  $b P_2 a$ . In both cases, we have  $f(P_1, P'_2, \mathbf{P}_{-\{1,2\}}) = b$  and  $f(P'_1, P'_2, \mathbf{P}_{-\{1,2\}}) = a$  by Lemma 4. If we repeat the same process for agent 3, 4, ..., and  $n$ , then  $(P_1, \mathbf{P}'_{-1})$  and  $(P'_1, \mathbf{P}'_{-1})$  are top-2 profiles but do not satisfy elementary monotonicity. This contradicts the weak elementary monotonicity of  $f$ . Next, Lemma 5 completes the proof:

**Lemma 5** (Mishra 2016). For any domain  $\mathcal{D} \subseteq \mathcal{P}$ , an scf  $f : \mathcal{D} \rightarrow A$  is LDSIC if and only if it is G-LOBIC and satisfies elementary monotonicity.

□

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